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THE TWO-PEBBLING PROPERTY ON SHADOW GRAPH OF A PATH

A. LOURDUSAMY[1]
Department of Mathematics, St. Xavier's College (Autonomous),
Palayamkottai - 627 002, Tamil Nadu, INDIA.

S. SARATHA NELLAINAYAKI[2]
Department of Mathematics, St. Xavier's College (Autonomous)
Palayamkottai - 627 002, Tamil Nadu, INDIA.

E-mail: [1]lourdusamy15@gmail.com, [2]sarathas1993@gmail.com

Abstract

Given a distribution of pebbles on the vertices of a connected graph G, a pebbling move is defined as the removal of two pebbles from some vertex and the placement of one of those pebbles at an adjacent vertex. The pebbling number, $f(G)$, of a connected graph G, is the smallest positive integer such that from every placement of $f(G)$ pebbles, we can move a pebble to any specified vertex by a sequence of pebbling moves. A graph G has the 2-pebbling property if for any distribution with more than $2f(G)-q$ pebbles, where q is the number of vertices with at least one pebble, it is possible, using the sequence of pebbling moves, to put 2 pebbles on any vertex. In this paper, we find the pebbling number for the shadow graph of a path and show that it satisfies the 2−pebbling property.

Keywords: Pebbling number, 2-pebbling property, Shadow graph.

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1. Introduction

Pebbling, one of the latest evolutions in graph theory proposed by Lakarias and Saks, has been the topic of vast investigation with significant observations. Having Chung [1] as the forerunner to familiarize pebbling into writings, many other authors too have developed this topic. Hulbert published a survey of graph pebbling [5].

Consider a connected graph with fixed number of pebbles distributed on its vertices. A pebbling move consists of the removal of two pebbles from a vertex and placement of one of those pebbles at an adjacent vertex. The pebbling number of a vertex v in a graph G is the smallest number $f(G,v)$ such that for every placement of $f(G,v)$ pebbles, it is possible to move a pebble to v by a sequence of pebbling moves. Then the pebbling number of G is the smallest number, $f(G)$ such that from any distribution of $f(G)$ pebbles, it is possible to move a pebble to any specified target vertex by a sequence of pebbling moves. Thus $f(G)$ is the maximum value of $f(G,v)$ over all vertices v.

Chung [1] defined the 2-pebbling property of a graph. Given a distribution of pebbles on G, let p be the number of pebbles, q be the number of vertices with at least one pebble, we say that G satisfies the 2 - pebbling property, if it is possible to move two pebbles to any specified vertex whenever p and q satisfy the inequality $p+q>2f\left(G\right)$.

In this paper, we find the pebbling number for the shadow graph of a path and show that the 2-pebbling property. In Section 2, we give some useful results for the subsequent sections. In Section 3, we determine the pebbling number for the shadow graph of a path $D_2(P_n)$. In Section 4, we prove that the shadow graph of a path $D_2(P_n)$ satisfies the 2-pebbling property.

2. Preliminary

We now introduce some definitions and notations which will be useful for the subsequent sections. For graph theoretic terminologies we refer to [4].

Definition 2.1. The *shadow graph* $D_2(G)$ of a connected graph G is constructed by taking two copies of G, say G_1 and G_2 and joining each vertex u in G_1 to the neighbours of the corresponding vertex v in G_2 .

The shadow graph of a path is denoted by $D_2(P_n)$. Label the vertices in the first copy of the path by $x_1, x_2, \ldots, x_n$ and the vertices in the second copy of the path by $x_{n+1}, x_{n+2}, \ldots, x_{2n}$ starting from the left.

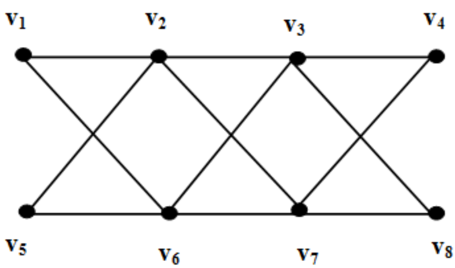

Figure 1.1. $D_2(P_n)$

Theorem 2.2 *[3] Let P_n be a path on n vertices. Then $f(P_n) = 2^{n-1}$.*

Theorem 2.3. *[2] Let $K_{1,n}$ be a star graph, where n > 1. Then $f\left(K_{1,n}\right) = n + 2$.*

Theorem 2.4. *[6] (i). Let C_{2k} be an even cycle on 2k vertices. Then $f(C_{2k}) = 2^k$.*

(ii) Let C_{2k+1} be an odd cycle on 2k + 1 vertices. Then

$$f(\mathrm{C}_{2k+1}) = 2\left\lfloor \frac{2^{k+1}}{3} \right\rfloor + 1.$$

Theorem 2.5. *[6] Let G be a graph with diameter G = 2. Then G has the 2- pebbling property.*

Theorem 2.6. *[3] All paths satisfy the 2-pebbling property.*

Theorem 2.7. *[3] All cycles have the 2-pebbling property.*

3. Pebbling on the Shadow graph of a path $D_2(P_n)$

Remark 3.1. A distribution of pebbles on the vertices of the graph G is a function $p : V(G) \to N \cup \{0\}$. Let $p(v)$ denote the number of pebbles on the vertex v and $p(\mathrm{A})$ denote the number of pebbles on the vertices of the set $A \subseteq V(G)$. Let v be a target vertex in the graph G. If $p(v) = 1$ or $p(u) \geq 2$, where $uv \in E(G)$, then we can move a pebble to v easily. So we always assume that $p(v) = 0$ and $p(u) \leq 1$ for all $uv \in E(G)$, when v is the target vertex.

For $n = 2$, $D_2(P_2)$ is isomorphic to C_4, we have the following theorem.

Theorem 3.2. *[3] For the shadow graph of the path* P_2, $f(D_2(P_2)) = 4.$

Theorem 3.3. *For the shadow graph of the path* P_3, $f(D_2(P_3)) = 6.$

Proof. Placing 3 pebbles on x_1 and one pebble on each x_3 and x_4, we cannot reach x_6. Thus $f(D_2(P_3)) \geq 6.$

Now we prove that $f(D_2(P_3)) \leq 6.$ Let D be any distribution of 6 pebbles on the vertices of $D_2(P_3)$.

***Case* 1 :** Let x_3 be the target vertex.

Clearly $p(x_3) = 0,$ $p(x_2) \leq 1$ and $p(x_6) \leq 1$ by Remark 3.1. If $p(x_2) = 1$ we can move another pebble to x_2, since x_1 or x_4 or x_6 contains at least 2 pebbles. Therefore assume that $p(x_2) = 0.$ Since $< D_2(P_3) - \{x_2\} >$ is isomorphic to $K_{1,4}$ and $f(K_{1,4}) = 6,$ we are done. Similarly, we are done if x_1, x_4 and x_6 are the target vertices.

***Case* 2 :** Let x_2 be the target vertex.

Clearly $p(x_2) = 0,$ $p(x_1) \leq 1,$ $p(x_3) \leq 1,$ $p(x_4) \leq 1$ and $p(x_6) \leq 1.$ Suppose $p(x_i) = 1$ for some $i \in \{1, 3, 4, 6\}$. Then using the path $P : x_5 x_i x_2$ we can pebble the target. Suppose $p(x_i) = 0$ for every $i \in \{1, 3, 4, 6\}$. Then $p(x_5) \geq 5$ and hence we are done. Similarly, we are done if x_5 is the target vertex.

Theorem 3.4. *For the shadow graph of a path* P_n, $f(D_2(P_n)) = 2^{n-1} + 2$, *(n≥4).*

Proof. Placing 2^{n-1} pebbles on x_1 and one pebble on each x_n and x_{n+1}, we cannot reach x_{2n}. Thus $f(D_2(P_n)) \geq 2^{n-1} + 2$. Now we prove the sufficient part by induction on n. Clearly, it is true for n = 3, by Theorem 3.3. So, we assume the result is true for $4 \leq n' < n$. Let D be any distribution of $2^{n-1} + 2$ pebbles on the vertices of $D_2(P_n)$.

***Case* 1 :** Let v be any target vertex other than x_1, x_n, x_{n+1} and x_{2n}.

Clearly, $p(v) = 0$ by Remark 3.1. We note that $d(u, x_1) \leq n-2$ and $d(u, x_{n+1}) \leq n-2$ for every $u \notin \{x_n, x_{2n}\}$. If $p(x_1) \geq 2^{n-2}$ or $p(x_{n+1}) \geq 2^{n-2}$, then we are done. Therefore assume that $p(x_1) \leq 2^{n-2} - 1$ and $p(x_{n+1}) \leq 2^{n-2} - 1$. By moving as many pebbles as possible from x_1 to x_2 and from x_{n+1} to x_{n+2}, we see that the subgraph $< D_2(P_n) - \{x_1, x_{n+1}\} > \cong D_2(P_{n-1})$ contains at least $2^{n-2} + 2$ pebbles and hence we are done, by induction.

***Case* 2 :** Let v be any target vertex, where $v \in \{x_1, x_n, x_{n+1}, x_{2n}\}$.

Without loss of generality, we assume that $v = x_1$. Let $p(x_2) + p(x_3) + \ldots + p(x_{n-1}) + p(x_{n+1}) + \ldots + p(x_{2n-1}) = p$.

Suppose $p \geq 3$. Then $p(x_n)$ + $p(x_{2n})$= $2^{n-1} + 2 - p \leq 2^{n-1} - 1$. After moving as many pebbles as possible from x_n to x_{n-1} and from x_{2n} to x_{2n-1}, we see that $< D_2(P_n) - \{x_n, x_{2n}\} > \cong D_2(P_{n-1})$ contains at least $2^{n-2}+2$ pebbles and hence we are done, by induction. Therefore assume that $p \leq 2$.

***Subcase* 2.1 :** Let $p = 0$.

Clearly, $p(x_n) + p(x_{2n}) = 2^{n-1} + 2$. After moving as many pebbles as possible from x_n to x_{n-1} and from x_{2n} to x_{n-1}, the vertex x_{n-1} contains at least 2^{n-2} pebbles. Since $d(x_1, x_{n-1}) = n-2$, we are done.

***Subcase* 2.2 :** Let $p = 1$.

Now $p(x_n) + p(x_{2n}) = 2^{n-1} + 1$. After moving as many pebbles as possible from x_n to x_{n-1} and from x_{2n} to x_{n-1}, the vertex x_{n-1} contains at least $\left\lfloor \frac{2^{n-1}+1-1}{2} \right\rfloor = 2^{n-2}$ pebbles. Since $d(x_1, x_{n-1}) = n-2$, we are done.

***Subcase* 2.3 :** Let $p = 2$.

Now $p(x_n) + p(x_{2n}) = 2^{n-1}$. Then we can move at least $\left\lfloor \frac{2^{n-1}-2}{2} \right\rfloor = 2^{n-2} - 1$ pebbles to x_{n-1} or x_{2n-1}. Hence we can reach the vertex x_2. If $p(x_{n+1}) = 2$ we can move one pebble to x_2 and hence we reach the target. Therefore assume that $p(x_{n+1}) \leq 1$. Since $p = 2$, there exists an j such that the vertex x_j is occupied, where $j \in \{2, 3, \ldots n-1, n+2, \ldots, 2n-1\}$ and hence we can easily reach the target.

4. The 2-pebbling property

In this section, we show that the shadow graph of a path $D_2(P_n)$ satisfies the 2-pebbling property. Since $D_2(P_2)$ is isomorphic to C_4 and by Theorem 2.7, $D_2(P_2)$ has the 2 - pebbling property.

Remark 4.1. Consider the graph G with n vertices and $2f(G)-q+1$ pebbles on it and we choose a target vertex v from G. If $p(v) = 1$, then the number of pebbles remaining in G is $2f(G)-q \geq f(G)$, since $f(G) \geq n$ and $q \leq n$, and hence we can move the second pebble to v. Let us assume that $p(v)=0$. If $p(u) \geq 2$, where $uv \in E(G)$, we move a pebble to v from u. Then the graph G has at least $2f(G)-q+1-2$ pebbles, since $f(G) \geq n$ and $q \leq n-1$, and hence we can move the second pebble to v. So, we always assume that $p(v)=0$ and $p(u) \leq 1$ for all $uv \in E(G)$, when v is the target vertex.

Theorem 4.2. *The graph $D_2(P_3)$ satisfies the two-pebbling property.*

Proof. Since the diameter of $D_2(P_3)$ is two, by Theorem 2.5 we conclude that, the graph $D_2(P_3)$ satisfies the two-pebbling property.

We first prove the following lemma.

Lemma 4.3. *Given any distribution of $2f(D_2(P_n))-q+1$ pebbles on the vertices of $D_2(P_n)$, $n \geq 3$, we can move two pebbles to x_1 and retain at least two* *pebbles on* $D_2(P_n)$. *If* $p(x_{n+1}) = 0$ *we can move two pebbles to* x_1 *and retain at least* 3 *pebbles on* $D_2(P_n)$.

Proof. The proof is by induction on n. Let n = 3. Let D be any distribution of 2(6) − q + 1 = 13 − q pebbles on it. Clearly $p(x_1)$ = 0, $p(x_2) \le 1$ and $p(x_5) \le 1$ by Remark 4.1. Then $q \le 5$.

Suppose q = 5. Now 8 pebbles are distributed on the vertices of the graph $D_2(P_3)$. Also $p(x_1)$ = 0, $p(x_2)$ = 1 and $p(x_5)$ = 1. Thus we can easily move 2 pebbles to x_1 using 6 pebbles and hence we can retain 2 pebbles. Therefore assume that $q \le 4$. Then either x_2 or x_5 is occupied.

Without loss of generality, assume $p(x_2)$ = 1. Then using 2 pebbles we can move an additional pebble to x_2 and hence we can move a pebble to x_1. Now the remaining number of pebbles distributed on the vertices of $D_2(P_3)$ is at least $13 - q - 3 \ge 10 - q \ge 6$, since $q \le 4$. By Theorem 2.3, we can move an additional pebble to x_1 at a cost of at most 2^2 = 4 pebbles and hence we can retain 2 pebbles. Similarly we are done, if $p(x_2)$ = 0 and $p(x_5)$ = 1. Therefore assume that $p(x_2)$ = 0 = $p(x_5)$.

If $p(x_4) \ge 2$, then one pebble can be moved to x_2. Using 2 pebbles we can move an additional pebble to x_2 and thus we can move a pebble to x_1. Now the remaining number of pebbles distributed on the vertices of $D_2(P_3)$ is at least $13-q-4=9-q\ge 6$, since

$q \leq 3$. Then by Theorem 3.3, we can move a pebble to x_1 at a cost of at most 4 pebbles and hence we can retain at least 2 pebbles. Therefore assume that $p(x_4) \leq 1$.

Suppose $p(x_4) = 0$. Then $p(x_3) + p(x_6) = 13 - q \geq 11$. Clearly using 8 pebbles we can move 2 pebbles to x_1 and retain at least 3 pebbles. Therefore assume $p(x_4) = 1$. Then $p(x_3) + p(x_6) = 13 - q \geq 10$. Again using 8 pebbles we can move 2 pebbles to x_1 and retain at least 2 pebbles. Hence the lemma holds, when $n = 3$.

Assume the Lemma is true for $4 \leq n' < n$. Let D be any distribution of $2(2^{n-1}+2)-q+1$ pebbles on the vertices of $D_2(P_n)$. Clearly, $p(x_1) = 0$, $p(x_2) \leq 1$ and $p(x_{n+2}) \leq 1$ by Remark 4.1. Suppose $q = 2n - 1$. Then $p(x_2) = 1$ and $p(x_{n+2}) = 1$. Using $2n$ pebbles, we can easily put 2 pebbles to x_1 and the remaining number of pebbles is

$$2(2^{n-1}+2)-q+1-2n = 2^n+6-4n = 2+(2^n+4-4n).$$

Since $n \geq 4$, $2(2^{n-1}+2)-q+1-2n \geq 2$. Hence we can retain at least 2 pebbles on $D_2(P_n)$. Therefore assume that $q \leq 2n - 2$. Then either x_2 or x_{n+2} is occupied. Without loss of generality, assume $p(x_2)=1$ and $p(x_{n+2})=0$. We can move a pebble to x_2 at a cost of at most 2^{n-2} pebbles and hence we can move a pebble to x_1. Now the remaining number of pebbles is at least

$$2(2^{n-1}+2)-q+1-2^{n-2}-1=2^{n-1}+2+2^{n-2}+2-q.$$

Since $q \leq 2n-2$ and $n \geq 4$, we conclude that

$$2(2^{n-1}+2)-q+1-2^{n-2}-1 \geq 2^{n-1}+2=f(D_2(P_n)).$$

By Theorem 3.4, we can move a pebble to x_1 at a cost of at most 2^{n-1} pebbles and retain at least 2 pebbles. Similarly we are done, if $p(x_2)=0$ and $p(x_{n+2})=1$. Therefore assume $p(x_2)=0$ and $p(x_{n+2})=0$. If $p(x_{n+1}) \geq 2$, one pebble can be moved to x_2. And we can move an additional pebble to x_2 at a cost of at most 2^{n-2} pebbles, since $dist(x_2,u) \leq n-2$, for every $u \in V(D_2(P_n))$. Thus we can put 2 pebbles to x_2 and hence a pebble to x_1. Now the remaining number of pebbles is at least

$$2(2^{n-1}+2)-q+1-2^{n-2}-2 \geq 2^{n-1}+2=f(D_2(P_n)),$$

since $q \leq 2n-3$. By Theorem 3.4, we can move a pebble to x_1 at a cost of at most 2^{n-1} pebbles and retain at least 2 pebbles. Therefore assume $p(x_{n+1}) \leq 1$.

***Case* 1:** Let $p\left(x_{n+1}\right)=0.$

In this case, $2(2^{n-1}+2)-q+1$ pebbles are distributed on the vertices of $< D_2(P_n)-\{x_1,x_2,x_{n+1},x_{n+2}\} >$. Thus using $2(2^{n-2}+2)-q+1$ pebbles, we can move 2 pebbles to x_2 and retain at least 3 pebbles, since $p(x_{n+2})=0$ and by induction. Therefore one pebble can be moved to x_1. Now the remaining number of pebbles is at least

$$2(2^{n-1}+2)-q+1-[2(2^{n-2}+2)-q+1]+3=2^{n-1}+2+1=f(D_2(P_n))+1.$$

By Theorem 3.4, we can move a pebble to x_1 at a cost of at most 2^{n-1} pebbles and retain at least 3 pebbles.

***Case* 2:** Let $p\left(x_{n+1}\right)=1.$

Now using $2(2^{n-2}+2)-(q-1)+1$ pebbles, we can move 2 pebbles to x_2 and retain at least 3 pebbles, since $p(x_{n+2})=0$ and by induction. Therefore one pebble can be moved to x_1. Now the remaining number of pebbles is at least

$$2(2^{n-1}+2)-q+1-[2(2^{n-2}+2)-(q-1)+1]+3=2^{n-1}+2=f(D_2(P_n)).$$

By Theorem 3.4, we can move a pebble to x_1 at a cost of at most 2^{n-1} pebbles and retain at least 2 pebbles on $D_2(P_n)$.

Theorem 4.4. *The graph* $D_2(P_n)$ *satisfies the two-pebbling property.*

Proof. The proof is by induction on n. Clearly, the result is true for n = 3 by Theorem 4.2. Assume the theorem is true for $4\le n'<n$.

Let D be any distribution of $2f(D_2(P_n))-q+1$ $=2(2^{n-1}+2)-q+1$ pebbles on the vertices of $D_2(P_n)$.

***Case* 1 :** Let x_1 be the target vertex.

By Lemma 4.3, we can move 2 pebbles to x_1. By symmetry we are done, if x_n, x_{n+1} and x_{2n} are the target vertices.

***Case* 2 :** Let x_2 be the target vertex.

Clearly $p(x_2)=0$, $p(x_1)\le 1$, $p(x_3)\le 1$, $p(x_{n+1})\le 1$ and $p(x_{n+2})\le 1$ by Remark 4.1. Now the number of pebbles distributed on the vertices of $< D_2(P_n)-\{x_1,x_{n+1}\} >$ is at least $2(2^{n-1}+2)-q+1-2$ $\ge 2(2^{n-2}+2)-q_1+1$, where q_1 is the number of occupied vertices in $< D_2(P_n)-\{x_1,x_{n+1}\} >$. Clearly $q\ge q_1$. Thus we can move 2 pebbles to x_2 by induction. By symmetry we are done, if x_{n-1}, x_{n+2} and x_{2n-1} are the target vertices.

***Case* 3 :** Let x be the target vertex, other than x_1, x_2, $x_{n-1}, x_n, x_{n+1}, x_{n+2}$, x_{2n-1} and x_{2n}.

If $p(x_1)+p(x_{n+1})\ge 2^{n-2}+1$, then we can move as many pebbles as possible from x_1 and x_{n+1} to x_2 and hence by using the path $P: x_2x_3\cdots x$, we are done.

Suppose $p(x_1)+p(x_{n+1})\le 2^{n-2}$. Then $< D_2(P_n)-\{x_1,x_{n+1}\} >$ has at least $2(2^{n-1}+2)-q+1-2^{n-2}\ge 2(2^{n-2}+2)-q_1+1$, where q_1 is the number of occupied vertices in $< D_2(P_n)-\{x_1,x_{n+1}\} >$, since $q>q_1$. Thus we can move 2 pebbles to x by induction.

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